A Gateway to All Post Graduate Courses

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Production of Courseware

-Content for Post Graduate Courses

Paper No. : Electromagnetic Theory

Module : Maxwell's Equations - II

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Description of Module	
Subject Name	Physics
Paper Name	Electromagnetic Theory
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Contents

Learning Objectives:

From this module students may get to know about the following:

1. The transformations of the potentials, called gauge transformations, that leave the fields unchanged and decouple the equations. The decoupled equations take the form of the wave equation in three dimensions.
2. The green's function method of solving inhomogeneous differential equations.
3. The use of the Green's function method to solve the inhomogeneous wave equation.

1. Gauge transformations

This module is essentially a continuation of module 3 and here we complete the job left undone in it. In Maxwell's equations – I, unit 3, by introducing the potentials, $(\Phi, \vec{A})$, we had reduced the set of four Maxwell's equations into a set of two equations

$$\nabla^2\Phi + \frac{\partial}{\partial t}(\vec{\nabla}.\vec{A}) = -\rho/\varepsilon_0$$

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} - \vec{\nabla}.(\vec{\nabla}.\vec{A} + \frac{1}{c^2}\frac{\partial\Phi}{\partial t}) = -\mu_0\vec{J}$$

where

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

$$\vec{E} = -\vec{\nabla}\Phi - \frac{\partial\vec{A}}{\partial t}$$

However in a sense we have not achieved much. Instead of being first order they are now second order differential equations, and more importantly, they are still coupled. What is important is to somehow decouple the equations so that they can be solved independently. This is achieved by exploiting the arbitrariness in the definition of potentials. Since curl of gradient is zero, we can always add gradient of a scalar to $\vec{A}$ without changing anything. In other words, $\vec{B}$ remains unchanged under the transformation

$$\vec{A} \to \vec{A}' = \vec{A} + \vec{\nabla}\Lambda\,. \qquad (2)$$

However, $\vec{E}$ in general will change under this transformation unless Φ is also transformed simultaneously to

$$\Phi \to \Phi' = \Phi - \frac{\partial\Lambda}{\partial t}\,. \qquad (3)$$

The set of transformations represented by equations (2) and (3) under which the electric and magnetic fields remain unchanged are called gauge transformations. This allows us certain freedom in the choice of the potentials ($\vec{A},\Phi$). If we are not happy with one set, we can always choose another via the gauge transformations which may be more convenient for solving the set of equations. This freedom of the choice of gauge can be exploited to obtain various sets of gauges which are all useful in different contexts. We will look at two gauges which are the ones most commonly employed.

2.1 Lorentz gauge

In the Lorentz gauge, the freedom implied by equation (2) and (3) is used to choose ($\vec{A},\Phi$) such that

$$\vec{\nabla}.\vec{A}+\frac{1}{c^2}\frac{\partial\Phi}{\partial t}=0. \tag{25}$$

With this auxiliary condition, the equations (21) and (22) for the potentials take the form:

$$\nabla^2\Phi-\frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2}=-\rho/\varepsilon_0 \tag{26}$$

$$\nabla^2\vec{A}-\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}=-\mu_0\vec{J} \tag{27}$$

The equations are now well and truly decoupled. Not only that, they both have the same form – that of the well-known wave equation. It is thus clear that the equations admit of wave like solutions for ($\vec{A},\Phi$) which leads to wave-like solutions for ($\vec{E},\vec{B}$) as well.

But first we have to ensure that potentials can always be found which satisfy the Lorentz gauge condition (25). Suppose that potentials ($\vec{A},\Phi$) that satisfy (21) and (22) do not satisfy the gauge condition (25). Then we can make a gauge transformation to potentials ($\vec{A}',\Phi'$) and demand that potentials ($\vec{A}',\Phi'$) satisfy (25):

$$\vec{\nabla}.\vec{A}'+\frac{1}{c^2}\frac{\partial\Phi'}{\partial t}=0.$$

On using (16) and (17), this implies

$$\vec{\nabla}.\vec{A}+\frac{1}{c^2}\frac{\partial\Phi}{\partial t}+\nabla^2\Lambda-\frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t^2}=0$$

In other words, all we have to do is to find any scalar function Λ that satisfies the equation

$$\nabla^2\Lambda-\frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t^2}=-(\vec{\nabla}.\vec{A}+\frac{1}{c^2}\frac{\partial\Phi}{\partial t})$$

use it to transform from ($\vec{A},\Phi$) to ($\vec{A}',\Phi'$), then ($\vec{A}',\Phi'$) do satisfy the gauge condition. Interestingly, the gauge function Λ that permits the Lorentz condition to be satisfied, is itself the solution of a wave equation.

2.1 Restricted gauge transformations

Even for potentials that satisfy a particular gauge condition there is further arbitrariness. If we make another gauge transformation

$$\vec{A}\rightarrow\vec{A}'=\vec{A}+\vec{\nabla}\Lambda$$

$$\Phi \rightarrow \Phi' = \Phi - \frac{\partial \Lambda}{\partial t}$$

where

$$\nabla^2 \Lambda - \frac{1}{c^2}\frac{\partial^2 \Lambda}{\partial t^2} = 0$$

the new potentials will also satisfy the Lorentz gauge condition. Such transformations within a gauge are called restricted gauge transformations. Not only we are assured that solutions that satisfy the Lorentz gauge condition exist, we have found that there are infinity of them connected to each other by restricted gauge transformations.

In the Lorentz gauge everything is a wave. The potentials, the derived fields and the scalar gauge field are all solutions of wave equations. This is also the gauge in which the equations of electrodynamics can be written in an explicitly covariant form in special relativity.

2.3 Coulomb gauge

Another useful gauge for the potentials is the Coulomb gauge for which the gauge condition is

$$\vec{\nabla}.\vec{A} = 0 . \qquad (28)$$

For this gauge, equation (21) simply becomes

$$\vec{\nabla}^2 \Phi(\vec{x},t) = -\rho(\vec{x},t)/\varepsilon_0 , \qquad (29)$$

where we have shown the space-time dependence explicitly. The scalar potential Φ is thus a solution of the Poisson equation. In other words, it is just the instantaneous Coulomb potential

$$\Phi(\vec{x},t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{x}',t)}{|\vec{x}-\vec{x}'|}d^3x'; \qquad (30)$$

Hence the name, "Coulomb gauge". Before proceeding further, let us first demonstrate, as before, that potentials that satisfy the Coulomb gauge can always be found. If the potentials ($\vec{A},\Phi$) that satisfy equations (21) and (22) do not satisfy the gauge condition (28), we can make a gauge transformation to potentials ($\vec{A}',\Phi'$) and demand that potentials ($\vec{A}',\Phi'$) satisfy (28):

$$\vec{\nabla}.\vec{A}' = 0.$$

On using equation (16) this implies

$$\vec{\nabla}.\vec{A} + \nabla^2\Lambda = 0$$

In other words, all we have to do is to find any scalar function Λ that satisfies the Poisson equation

$$\nabla^2\Lambda = -\vec{\nabla}.\vec{A}.$$

In Coulomb gauge, the equation for $\vec{A}$ takes the form

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J} + \frac{1}{c^2}\vec{\nabla}\frac{\partial\Phi}{\partial t} \qquad (31)$$

This has the form of a wave equation for $\vec{A}$ but is rather awkward since it contains Φ as well, though Φ is obtained from solution of Poisson equation. However, the equations can be decoupled explicitly in the Coulomb gauge as well. Given any vector field, $\vec{J}$, it can always be decomposed into two parts, the transverse part $\vec{J}_t$ and a longitudinal part $\vec{J}_l$, such that

$$\vec{\nabla}.\vec{J}_t = 0;$$

$$\vec{\nabla}\times\vec{J}_l = 0;$$

with

$$\vec{J} = \vec{J}_t + \vec{J}_l\,.$$

The longitudinal part is given by

$$\vec{J}_l = -\frac{1}{4\pi}\vec{\nabla}\int\frac{\vec{\nabla}'.\vec{J}(\vec{x}',t)}{|\vec{x}-\vec{x}'|}d^3x'$$

and the transverse part by

$$\vec{J}_t = \frac{1}{4\pi}\vec{\nabla}\times(\vec{\nabla}\times\int\frac{\vec{J}(\vec{x}',t)}{|\vec{x}-\vec{x}'|}d^3x')$$

Since $\vec{J}_l$ is the gradient of a scalar [$\int\frac{\vec{\nabla}'.\vec{J}(\vec{x}',t)}{|\vec{x}-\vec{x}'|}d^3x'$], and $\vec{J}_t$ is the curl of a vector [$(\vec{\nabla}\times\int\frac{\vec{J}(\vec{x}',t)}{|\vec{x}-\vec{x}'|}d^3x')$], it follows that

$$\vec{\nabla}\times\vec{J}_l = 0$$

and

$$\vec{\nabla}.\vec{J}_t = 0$$

as required. All that is left to be shown is that the sum of the two is indeed equal to $\vec{J}$. Using the vector relation

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}.\vec{A}) - \nabla^2\vec{A}$$

for any vector $\vec{A}$, we can write

$$\vec{J}_t = \frac{1}{4\pi}\vec{\nabla}\times(\vec{\nabla}\times\int\frac{\vec{J}(\vec{x}',t)}{|\vec{x}-\vec{x}'|}d^3x')$$

$$= \frac{1}{4\pi}\vec{\nabla}(\vec{\nabla}.\int\frac{\vec{J}(\vec{x}',t)}{|\vec{x}-\vec{x}'|}d^3x') - \frac{1}{4\pi}\nabla^2\int\frac{\vec{J}(\vec{x}',t)}{|\vec{x}-\vec{x}'|}d^3x'.$$

Let us look at the first term of $\vec{J}_t$:

$$\frac{1}{4\pi}\vec{\nabla}(\vec{\nabla}.\int\frac{\vec{J}(\vec{x}',t)}{|\vec{x}-\vec{x}'|}d^3x') = \frac{1}{4\pi}\vec{\nabla}\int\vec{J}(\vec{x}',t).\vec{\nabla}\frac{1}{|\vec{x}-\vec{x}'|}d^3x'$$

$$= -\frac{1}{4\pi}\vec{\nabla}\int\vec{J}(\vec{x}',t).\vec{\nabla}'\frac{1}{|\vec{x}-\vec{x}'|}d^3x'$$

$$= -\frac{1}{4\pi}\vec{\nabla}[\int\vec{\nabla}'.\frac{\vec{J}(\vec{x}',t)}{|\vec{x}-\vec{x}'|}d^3x' - \int\frac{\vec{\nabla}'.\vec{J}(\vec{x}',t)}{|\vec{x}-\vec{x}'|}d^3x']$$

Here $\vec{\nabla}'$ denotes differentiation with respect to $\vec{x}'$. Using the Gauss theorem, the first term above can be converted into a surface integral and assuming, as we usually do, that the fields fall of sufficiently rapidly, becomes zero. The second term is nothing but $-\vec{J}_l$. Hence

$$\vec{J}_t = -\vec{J}_l - \frac{1}{4\pi}\nabla^2 \int \frac{\vec{J}(\vec{x}',t)}{|\vec{x}-\vec{x}'|} d^3x'$$

or

$$\vec{J}_t + \vec{J}_l = -\frac{1}{4\pi}\nabla^2 \int \frac{\vec{J}(\vec{x}',t)}{|\vec{x}-\vec{x}'|} d^3x' = -\frac{1}{4\pi}\int \vec{J}(\vec{x},t)\nabla^2 \frac{1}{|\vec{x}-\vec{x}'|} d^3x'.$$

Further using the properties of delta – function:

$$\nabla^2 \frac{1}{|\vec{x}-\vec{x}'|} = -4\pi\delta^{(3)}(\vec{x}-\vec{x}'),$$

we obtain

$$\vec{J}_t + \vec{J}_l = -\int \vec{J}(\vec{x},t)\delta^{(3)}(\vec{x}-\vec{x}')d^3x' = \vec{J}(\vec{x},t)$$

So far the result is true for any vector field. For the problem at hand, the longitudinal part of the current can be recast by using equation of continuity (1) and the expression (23) for Φ into the following form:

$$\vec{J}_l = -\mu_0 \vec{\nabla} \int \frac{\vec{\nabla}'.\vec{J}(\vec{x}',t)}{|\vec{x}-\vec{x}'|} d^3x' = \mu_0 \vec{\nabla} \int \frac{\partial\rho/\partial t}{|\vec{x}-\vec{x}'|} d^3x' = \mu_0 \vec{\nabla} \frac{\partial}{\partial t} \int \frac{\rho(\vec{x}',t)}{|\vec{x}-\vec{x}'|} d^3x'$$

$$= \mu_0 \varepsilon_0 \vec{\nabla} \frac{\partial}{\partial t} \Phi = \frac{1}{c^2} \vec{\nabla} \frac{\partial}{\partial t} \Phi .$$

Using this in equation (24), we finally obtain

$$\nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J} + \frac{1}{c^2} \vec{\nabla} \frac{\partial \Phi}{\partial t} = -\mu_0 (\vec{J}_l + \vec{J}_t) + \frac{1}{c^2} \vec{\nabla} \frac{\partial \Phi}{\partial t} = -\mu_0 \vec{J}_t$$

$$= -\frac{\mu_0}{4\pi} \vec{\nabla} \times (\vec{\nabla} \times \int \frac{\vec{J}(\vec{x}',t)}{|\vec{x}-\vec{x}'|} d^3x') .$$

Since in this gauge the vector potential is given in terms of the transverse part of the current only, it is also called transverse gauge. It is also sometimes called the radiation gauge as the transverse currents give rise to purely transverse radiation fields far from the sources. The static potential is present but does not give rise to radiation. As we will learn later, the Coulomb gauge condition cannot be written in an explicit covariant form.

Despite all these complications associated with this gauge, it is in fact quite useful. Once the current is decomposed into two parts, the actual equations to be solved are quite simple. Also, in free space, where there are no sources, $\Phi = 0$ and the fields $\vec{E}$ and $\vec{B}$ can be found from $\vec{A}$ alone:

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t}, \qquad \vec{B} = \vec{\nabla} \times \vec{A} .$$

This gauge is particularly useful in quantum electrodynamics, since it necessitates the quantization of only the vector potential.

The equations are now well and truly decoupled. Not only that, they both have the same form – that of the well-known wave equation. It is thus clear that the equations admit of wave like solutions for ($\vec{A},\Phi$) which leads to wave-like solutions for ($\vec{E},\vec{B}$) as well.

So our task now is to solve the inhomogeneous wave equation whose general form is

$$\nabla^2\Psi(\vec{x},t)-\frac{1}{c^2}\frac{\partial^2\Psi(\vec{x},t)}{\partial t^2}=-4\pi f(\vec{x},t) \qquad (1)$$

where $f(\vec{x},t)$ is a known source function.

2. Green's function

One of the general methods for solving such inhomogeneous equations is the method of Green's function. We first provide a brief introduction to this method before we apply it to find the solution of the wave equation.

2.1 Green's function in one dimension

Let us illustrate the method of Green's function in one dimension. A green's function, $G(x,s)$ of a linear differential operator, $L=L(x)$, is any solution of the equation

$$LG(x,s)=\delta(x-s), \qquad (2)$$

where δ is the Dirac-delta function. This property of a Green's function can be exploited to solve inhomogeneous equations of the form

$$Lu(x)=f(x). \qquad (3)$$

Loosely speaking, if such a function G can be found for the operator L, then if we multiply the equation (2) for the Green's function by f(s), and then perform integration in the s variable, we obtain:

$$\int LG(x,s)f(s)ds=\int\delta(x-s)f(s)ds=f(x)$$

The right-hand side is now given by equation (3) to be equal to L u(x). Thus:

$$Lu(x)=\int LG(x,s)f(s)ds$$

Because the operator L = L(x) is linear and acts on the variable x alone (not on the variable of integration s), we can take the operator L outside of the integration on the right-hand side, obtaining

$$Lu(x) = L\int G(x,s)\, f(s)ds$$

which suggests

$$u(x) = \int G(x,s)\, f(s)ds \qquad (4)$$

Thus, we can obtain the function u(x) through knowledge of the Green's function in equation (2) and the source term on the right-hand side in equation (3). In other words, the solution of equation (3), u(x), can be determined by the integration given in equation (4). Although f(x) is known, this integration cannot be performed unless G is also known. The problem now lies in finding the Green's function G that satisfies equation (2).

The Green's function to a given differential operator is not unique. If $G(x,s)$ is a valid solution, adding any solution of the corresponding homogenous equation provides another valid solution. The green's function is "fixed" by the requirement of the boundary conditions that the solution must obey. Thus green's function is a solution of the differential operator along with the boundary conditions.

The extension of the Green's function to higher dimensions is straightforward. In three dimensions, the Green's function $G(\vec{x},\vec{s})$, for the differential operator $L(\vec{x})$ is the solution of the equation

$$LG(\vec{x},\vec{s}) = \delta(\vec{x}-\vec{s})$$

where $\delta(\vec{x}-\vec{s})$ is the three-dimensional Dirac-delta function:

$$\delta(\vec{x}) = \delta(x)\delta(y)\delta(z)\,.$$

Once the Green's function is determined, the solution of the inhomogeneous equation

$$Lu(\vec{x}) = f(\vec{x})$$

is given by

$$u(\vec{x}) = \int G(\vec{x},\vec{s})\, f(\vec{s})d^3s$$

2.2 Green's function for the wave equation

After this brief introduction, let us find the Green's function for the wave equation.

$$\nabla^2\Psi(\vec{x},t)-\frac{1}{c^2}\frac{\partial^2\Psi(\vec{x},t)}{\partial t^2}=-4\pi f(\vec{x},t)$$

The function Ψ is a function of four variables, $(\vec{x},t)$. So we write the Green's function as the solution of

$$\nabla^2 G(\vec{x},t;\vec{x}',t')-\frac{1}{c^2}\frac{\partial^2 G(\vec{x},t;\vec{x}',t')}{\partial t^2}=-4\pi\delta(\vec{x}-\vec{x}')\,\delta(t-t')\,. \qquad (5)$$

The solution $\Psi(\vec{x},t)$ is given in terms of the Green's function by

$$\Psi(\vec{x},t)=\iint d^3x'dt'G(\vec{x},t;\vec{x}',t')\,f(\vec{x}',t') \qquad (6)$$

We consider the simple case of no boundary surfaces so that it is the "free" Green's function that we are looking for. First of all we remove the explicit time dependence of the equation through Fourier transform to frequency. If the Fourier transform of a function $f(t)$ is defined by

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(\omega)e^{-i\omega t}d\omega \qquad (7)$$

then the inverse Fourier transform has the representation

$$f(\omega)=\int_{-\infty}^{\infty} f(t)e^{i\omega t}dt \qquad (8)$$

In the present case, let

$$G(\vec{x},t;\vec{x}',t')=\frac{1}{2\pi}\int_{-\infty}^{\infty} G(\vec{x},\omega;\vec{x}',t')e^{-i\omega t}d\omega \qquad (9)$$

Substituting equation (9) in equation (5)

$$[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}]\frac{1}{2\pi}\int_{-\infty}^{\infty} G(\vec{x},\omega;\vec{x}',t')e^{-i\omega t}d\omega = -4\pi\delta(\vec{x}-\vec{x}')\delta(t-t')$$

Since time dependence appears only in the factor $e^{-i\omega t}$, and $\frac{\partial}{\partial t}e^{-i\omega t} = -i\omega e^{-i\omega t}$, the equation becomes

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}[\nabla^2 + \frac{\omega^2}{c^2}]G(\vec{x},\omega;\vec{x}',t')e^{-i\omega t}d\omega = -4\pi\delta(\vec{x}-\vec{x}')\delta(t-t')$$

Now on taking the inverse Fourier transform, we have

$$[\nabla^2 + \frac{\omega^2}{c^2}]G(\vec{x},\omega;\vec{x}',t') = -4\pi\int_{-\infty}^{\infty}\delta(\vec{x}-\vec{x}')\delta(t-t')e^{i\omega t}dt = -4\pi\delta(\vec{x}-\vec{x}')e^{i\omega t'}$$

So let

$$G(\vec{x},\omega;\vec{x}',t') = g(\vec{x},\vec{x}')e^{iwt'} \qquad (10)$$

The Green's function $g(\vec{x},\vec{x}')$ satisfies the equation

$$[\nabla^2 + \frac{\omega^2}{c^2}]g(\vec{x},\vec{x}') = -4\pi\delta(\vec{x}-\vec{x}') \qquad (11)$$

The “sensible” boundary condition that we impose is $G_k(\vec{x},\vec{x}') \to 0$ as $|\vec{x}-\vec{x}'| \to \infty$. In other words, at a long distance from the source, the field goes to zero. Since the system we are solving is spherically symmetric about the point $\vec{x}'$, it is reasonable that the Green’s function itself be spherically symmetric, i.e., it is a function of only $|\vec{x}-\vec{x}'|$. Putting

$$\vec{x}-\vec{x}'=\vec{R}$$

the above equation reduces to

$$(\nabla^2+\frac{\omega^2}{c^2})g(R) = -4\pi\delta(\vec{R}) \qquad (12)$$

In spherical polar coordinates $\nabla^2\psi(\vec{r})$, the Laplacian of $\psi(\vec{r})$, has the form

$$\nabla^2\psi(\vec{r}) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial\psi}{\partial r}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\psi}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

For a function of r only, it reduces to

$$\nabla^2\Psi = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r})\Psi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Psi)$$

On using this expression for the Laplacian, equation (10) reduces to

$$\frac{1}{R}\frac{d^2}{dR^2}(Rg) + \frac{\omega^2}{c^2}g(R) = -4\pi\delta(\vec{R}) \,. \qquad (13)$$

Now $\delta(\vec{R}) = 0$ every where except R=0. Hence, except R=0, the function Rg(R) satisfies the homogeneous equation

$$\frac{d^2}{dR^2}(Rg(R)) + \frac{\omega^2}{c^2}(Rg(R)) = 0 \qquad (14)$$

This equation has the general solution

$$Rg(R) = Ae^{ikR} + Be^{-ikR} \qquad (15)$$

where A and B are constants, and $k = \omega / c$. This is the solution in free space away from the source at R=0. We have to select the solution which matches with the solution provided by the source term. Now the delta function in equation (13) has an influence only at $R = 0$. But for $R \to 0$, $kR \to 0$ and thus equation (11) reduces to the Green's function for Poisson equation:

$$\nabla^2 g(\vec{x}, \vec{x}') = -4\pi\delta(\vec{x} - \vec{x}')$$

This is the equation for the electrostatic potential for a unit point source located at $\vec{x}'$. The solution to this, of course, is well known. In the absence of any boundaries, it is just $\frac{1}{|\vec{x} - \vec{x}'|} = \frac{1}{R}$. Hence the correct normalization for our solution to the Helmholtz equation is

$$g(R) = \frac{1}{R} \text{ for } kR \to 0$$

The general solution for the Green's function is thus

$$g(R) = g(\vec{x}, \vec{x}') = \frac{Ae^{ikR} + Be^{-ikR}}{R} \tag{16}$$

with

$$A + B = 1 \tag{17}$$

Hence from equation (10)

$$G(\vec{x}, \omega; \vec{x}', t') = g(\vec{x}, \vec{x}')e^{i\omega t'} = \frac{Ae^{ikR} + (1-A)e^{-ikR}}{R}e^{i\omega t'}$$

We now take the inverse Fourier transform of this equation to get

$$G(\vec{x}, t; \vec{x}', t') = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{R}[Ae^{i\omega R/c} + (1-A)e^{-i\omega R/c}]e^{i\omega t'}e^{-i\omega t}d\omega$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{R}[Ae^{i\omega\{t'-(t-R/c)\}} + (1-A)e^{i\omega\{t'-(t+R/c)\}}]d\omega$$

Remember the representation of the δ-function

$$\delta(t-t') = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\omega(t-t')}d\omega \tag{18}$$

Use it now in the above equation to obtain

$$G(\vec{x}, t; \vec{x}', t') = \frac{1}{R}[A\delta\{t'-(t-\frac{R}{c})\} + (1-A)\delta\{t'-(t+\frac{R}{c})\}] \tag{19}$$

Or more explicitly

$$G(\vec{x},t;\vec{x}',t') = \frac{1}{|\vec{x}-\vec{x}'|}[A\delta\{t'-(t-\frac{|\vec{x}-\vec{x}'|}{c})\} + (1-A)\delta\{t'-(t+\frac{|\vec{x}-\vec{x}'|}{c})\}]. \qquad (20)$$

The two Green's functions $G^{(\pm)}(\vec{x},t;\vec{x}',t')$ given by

$$G^{(\pm)}(\vec{x},t;\vec{x}',t') = \frac{1}{|\vec{x}-\vec{x}'|}\delta\{t'-(t\mp\frac{|\vec{x}-\vec{x}'|}{c})\} \qquad (21)$$

are both independent solutions of equation (5). They are respectively called the retarded and the advanced Green's functions. Both these solutions, $G^{(\pm)}(\vec{x},t;\vec{x}',t')$, represent the response to a point source at $(\vec{x}',t')$, i.e., the field produced at the point $(\vec{x},t)$ by the source at the point $(\vec{x}',t')$. According to the retarded Green's function $G^{+}(\vec{x},t;\vec{x}',t')$, the response consists of a spherical wave centered at $\vec{x}'$ which propagates forward in time. In order to reach the point $\vec{x}$ at time t it must have started at the retarded time

$$t_{ret} = t - \frac{|\vec{x}-\vec{x}'|}{c}. \qquad (22)$$

According to the advanced Green's function $G^{+}(\vec{x},t;\vec{x}',t')$, the response consists of a spherical wave centered at $\vec{x}'$ which propagates backward in time. In order to reach the point $\vec{x}$ at time t it must have started at the advanced time

$$t_{adv} = t + \frac{|\vec{x}-\vec{x}'|}{c}.$$

The solution to our wave equation (1) is obtained on substituting the above expression (21) for $G(\vec{x},t;\vec{x}',t,)$ in equation (6):

$$\Psi^{(\pm)}(\vec{x},t) = \iint d^3x'dt'G^{(\pm)}(\vec{x},t;\vec{x}',t') = \iint d^3x'dt'\frac{\delta(t'-(t\mp\frac{|\vec{x}-\vec{x}'|}{c})}{|\vec{x}-\vec{x}'|}f(\vec{x}',t')$$

$$= \int d^3x'\frac{f(\vec{x}',t\mp\frac{|\vec{x}-\vec{x}'|}{c})}{|\vec{x}-\vec{x}'|}$$

Clearly, the advanced potential is not consistent with our ideas about causality, which demand that an effect can never precede its cause in time.

We are able to find solutions of the inhomogeneous wave equation (1) which propagate backward in time because this equation is time symmetric (i.e., it is invariant under the transformation $t \to -t$).

Thus, the solution which is consistent with our experience is the retarded solution

$$\Psi^{+}(\vec{x},t) = \iint d^3x'dt' f(\vec{x}',t')G^{(\pm)}(\vec{x},t;\vec{x}',t')$$

$$= \int d^3x'\frac{f(\vec{x}',t-\frac{|\vec{x}-\vec{x}'|}{c})}{|\vec{x}-\vec{x}'|} = \int d^3x'\frac{[f(\vec{x}',t')]_{ret}}{|\vec{x}-\vec{x}'|} \qquad (23)$$

The square bracket $[\ ]_{ret}$ denotes that the term inside is to be evaluated at the retarded time.

Summary

1. In this module we have introduced the student to the very useful and important concept of gauge transformations. These are transformations of the potentials that leave the fields invariant and hence any set of transformed potentials can be used to find the fields
2. Two specific “gauges”, the Lorentz gauge and the Coulomb gauge are introduced which are both useful in the solution of Maxwell equations. This freedom is made use of to decouple the equations and put them in the form of wave equations in three dimensions.
3. In the Lorentz gauge the equations are easily decoupled. A detailed mathematical derivation of how the decoupling of equations is to be achieved in the Coulomb gauge is provided

The method of Green’s function for solving inhomogeneous differential equations is next discussed for one dimensional problems and then applied to solve the wave equation in three dimensions.